K-theory of locally finite graph C^*-algebras

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August 5, 2011

Abstract

We calculate the K-theory of Cuntz-Krieger algebras associated to locally finite infinite graphs via the Bass-Hashimoto operator. The formula we get express the Grothendieck group and Whitehead group in purely graph theoretic terms. We consider the category of finite (black-and-white, double) subgraphs with certain graph homomorphisms and construct a continuous functor to abelian groups. This allows to present K_0 as an inductive limit of K-groups of finite graphs, which are calculated in [3]. Then we specify this construction in the case of an infinite graph with finite Betti number and obtain a formula $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)+\gamma(E)}$, where $\beta(E)$ is the first Betti number and $\gamma(E)$ is the branching number of the graph E. We note, that in the infinite case the torsion part of the group, which is present in the case of finite graph, vanishes. The formula for the Whitehead group expresses the group only via the first Betti number: $K_1(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)}$. This allows to provide a contrexample to the fact, that $K_1(\mathcal{O}_E)$ is a torsion free part of $K_0(\mathcal{O}_E)$, which holds for finite graphs.

MSC: Primary: 05C50, 46L80, 16B50 Secondary: 46L35, 05C63.

1 Introduction

Let us consider first a planar non-directed graph $\hat{E}$, which might be infinite, have loops, multiple edges and sinks.

We impose one finiteness condition on the graph, namely, it should be *locally finite:* every vertex should be connected only with finitely many vertices by edges. Also we suppose $\hat{E}$ is a connected non-directed graph (=geometrically connected graph).

We consider the Cuntz-Krieger algebras of the Bass-Hashimoto operator associated to this graph (more precisely, its infinite, locally finite analogue). This operator (operator Φ_E defined below) was considered by Hashimoto [5] and Bass [2] and later studied in [3]. The algebra known as a *boundary operator algebra*, studied, for example, in [7, 8, 9] is Morita equivalent to the corresponding algebra associated to the Bass-Hashimoto operator. Namely, $\mathcal{O}_E \sim C^*(\delta\mathcal{F})/\Gamma$, where $\mathcal{F}$ is a universal covering tree of the graph E and Γ is a free group of rank β, where β is the first Betti number of E. Note here, that the Cuntz-Krieger algebra of the Bass-Hashimoto operator associated to a given graph E (which we denote here also $\mathcal{O}_E$) should not be mixed with the Cuntz-Krieger algebra of operator defined by the incidence matrix of the graph, as it is done, for example, in [1, 11]. These are two different ways to associate Cuntz-Krieger algebra to a graph, via different operators. Although operators are very similar, the behaviour of the algebra changes dramatically. For example, as it can be seen from [1], K_0 of algebras associated to finite graphs via incidence matrix is not defined by the first Betti number, as it is the case for algebras associated to a finite graph via the Bass-Hashimoto operator [3].

Our goal here is to calculate K-theory of Cuntz-Krieger algebra of the Bass-Hashimoto operator associated to an infinite, locally finite graph, purely in graph theoretic terms, as it was done in [3] in the case of finite graph.

For any locally finite graph which is connected we can define a first Betti number (cyclomatic number) extending the usual definition for a finite graph.

Definition 1.1. If $\widehat{E}$ is a finite geometrically connected graph, then the first Betti number of $\widehat{E}$ is

$$\beta(\widehat{E}) = d_1 - d_0 + 1,$$

where d_0 is the cardinality of the set of vertices and d_1 is a cardinality of the set of (geometric) edges.

Note that for a finite graph with m connected components it would be

$$\beta(\widehat{E}) = d_1 - d_0 + m.$$

This number determines the number of cycles in $\widehat{E}$.

Definition 1.2. If $\widehat{E}$ is a locally finite geometrically connected graph, then we define the first Betti number of $\widehat{E}$ as the limit of the sequence of Betti numbers of finite subgraphs $\widehat{E}_k$ of $\widehat{E}$ obtained in the following way: $\widehat{E}_0$ is an arbitrary connected finite subgraph of $\widehat{E}$ and for any n, $\widehat{E}_{n+1}$ is obtained from

$\widehat{E}_n$ by adding to $\widehat{E}_n$ all edges of $\widehat{E}$ connected to the vertices of $\widehat{E}_n$ (together with vertices on the other end of these edges). It will be a finite graph due to $\widehat{E}$ is locally finite.

Remark 1. This definition does not depend on the choice of the subgraph $\widehat{E}_0$, starting for the sequence $\{\widehat{E}_n\}$. Indeed, if you start from another graph $\widehat{E}'_0$, then at some step n, we will have the graph $\widehat{E}_0$ as a subgraph of $\widehat{E}'_n$ and $\widehat{E'}_0$ as a subgraph of $\widehat{E}_n$ (all vertices and edges will be 'eaten' due to connectedness of $\widehat{E}$). It follows that the sequences $\beta(\widehat{E}_n)$ and $\beta(\widehat{E'}_n)$ have the same limit: either stabilize on the same positive integer or both grow to infinity.

Remark 2. It is clear that the locally finite infinite graph with finite first Betti number, has a shape of finite graph (with the same first Betti number), with finite or infinite number of outgoing trees.

Now associate for convenience to any graph $\widehat{E}$ as above, an oriented 'double' graph $E = (E^0, E^1, s, r)$ with the set of vertices E^0, set of edges E^1 and maps s, r from E^1 to E^0 which determines the source and the range of an arrow respectively (*source* and *range* maps). The quiver E is obtained from $\widehat{E}$ by doubling edges of $\widehat{E}$, so that each non oriented edge of $\widehat{E}$ gives rise to the pair of edges of E, e and $\overline{e}$, equipped with opposite orientations.

For any finite graph one can associate a Cuntz–Krieger C^*-algebra $\mathcal{O}_E$, in a way it is done in [3]. Namely, there it is considered a Cuntz–Krieger C^*-algebra (as it is defined in the Cuntz–Krieger paper [4]) associated to the matrix A_E. The matrix A_E is obtained from the graph as a matrix of the following operator (homomorphism of countable direct sum $\mathbb{Z}^{(E^1)}$ of copies of $\mathbb{Z}$) written as follows in the basis labeled by the set E^1 of edges of E:

$$\Phi_E : \mathbb{Z}^{(E^1)} \to \mathbb{Z}^{(E^1)} : \quad e \mapsto -\overline{e} + \sum_{e' : r(e) = s(e')} e' \qquad (*)$$

This operator was considered in [5] and [2] in connection with the study of Ihara zeta function of a graph.

The entries of the matrix A_E are in $\{0, 1\}$. If the graph E is finite, then A_E is an $2n \times 2n$ matrix, where n is the number of geometric edges of the graph E (=the number of edges of E, considered as a non-oriented graph).

Denote by $\mathcal{O}_E$ the Cuntz–Krieger C^*-algebra associated to this matrix A_E as in [4]. That is, $\mathcal{O}_E$ is the C^*-algebra generated by $2n$ partial isometries $\{S_j\}_{j=1}^{2n}$ which act on a Hilbert space in such a way that their support projections $Q_i = S_i^* S_i$ and their range projections $P_i = S_i S_i^*$ satisfy the

relations

$$P_iP_j = 0 \ \text{ if } i \neq j \ \text{ and } \ Q_i = \sum_{j=1}^{2n} (A_E)_{ij} P_j \ \text{ for } \ 1 \leqslant j \leqslant 2n.$$

This definition of $\mathcal{O}_E$ surely has a sense for an arbitrary locally finite matrix A_E, since the relations contain still finite sums.

So, we have the following.

Definition 1.3. For an infinite, row finite graph $E = (E^0, E^1, s, r)$, its C^*-algebra $\mathcal{O}_E$ is generated by partial isometries $\{S_i : i \in E^1\}$ subject to the relations

$$S_i^* S_i = \sum_{j \in E^1} (A_E)_{ij} S_j S_j^*.$$

Note that there could be another way to associate a C^*-algebra to a graph, for example, the one which is considered in [1]. It should be distinguished from the described above. In [1] a C^*-algebra of the graph defined as a Cuntz-Krieger algebra of another operator associated to a graph. Namely the operator represented by the edges adjacency matrix of an oriented graph.

In the paper [3] there was obtained a formula for $K_0(\mathcal{O}_E)$ depending only on the first Betti number $\beta(E)$ of the graph E, for a finite graph, namely $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)} \oplus \mathbb{Z}/(\beta(E)-1)\mathbb{Z}$. As a consequence, $K_1(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)}$, as it is well known ([10], [3] in the finite graphs setting that K_1 is a torsion free part of K_0.

It was mentioned there that it would be interesting to extend these results to infinite, locally finite graphs.

We do this here when the first Betti number $\beta(E)$ of an infinite graph is finite, and show that in the infinite case K_0 does not have torsion. Moreover, in the infinite case the formula for K_0 involves not only the first Betti number, but also another combinatorial characteristic of the graph: the *brunching number*.

Definition 1.4. We say that the brunching number $\gamma(E)$ of an infinite locally finite graph E with finite Betti number $\beta(E)$ is a number of different infinite chains outgoing from the finite subgraph of E with the Betti number $\beta(E)$.

Theorem 1.5. *Let E be a locally finite connected graph with the finite first Betti number $\beta(E)$ and the brunching number $\gamma(E)$. Then $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)+\gamma(E)}$.*

We also calculate $K_1(\mathcal{O}_E)$ and express it in terms of the first Betti number. It turns out that it is not necessary a torsion free part of $K_0(\mathcal{O}_E)$ in the infinite graphs setting.

Theorem 1.6. *Let E be a locally finite connected graph with the finite first Betti number $\beta(E)$. Then $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)+\gamma(E)}$.*

2 Category of black-and-white double graphs and functor to abelian groups

It is known for finite graphs and row-finite graphs (see, for example, [4] and [6]), that $K_0(\mathcal{O}_E) = coker(Id - \Phi)$, where $\Phi : \mathbb{Z}^{(E^1)} \to \mathbb{Z}^{(E^1)}$ is the homomorphism of countable direct sum of copies of $\mathbb{Z}$, defined for the graph E by the formula $(*)$.

Let E be an infinite locally finite double graph as above. We define a category $\mathcal{E}$ of black-and-white 'subgraphs' of E with morphisms of them in the following way. The objects of $\mathcal{E}$ are finite subgraphs of E, but with edges of two types: black and white, and the property that together with any edge e of any color it contains an edge of opposite direction $\bar{e}$ of the same color. In other words, we obtain a finite black-and-white 'subgraphs' of E as follows. Choose an arbitrary finite set Ω of vertices of E. A graph will contain all edges starting and ending on vertices from Ω. If the edge starting(ending) on a vertex from Ω, but ending (starting) outside, it will be white, otherwise black.

Now we define the set of graph homomorphisms between those black-and-white double graphs. There is a homomorphism $f : E \to F$ if F contains all vertices of E, and all black and white edges of E, as black edges of F. White vertices of F are those which are starting at the ends of vertices of E. So, these homomorphism changes white edges to black and add, as white, new edges which are coming out from former white edges. Any finite composition of defined above elementary homomorphisms is also a homomorphism. These homomorphisms play the role analogous to 'complete graph homomorphisms' in [1], however they are defined differently in our case.

Proposition 2.1. Every locally finite, infinite graph E is a direct limit of a chain of finite graphs and homomorphisms in the category $\mathcal{E}$. Any finite subgraph of E can serve as a starting element of this chain.

Proof. Take an arbitrary finite subgraph E_0 of E (as a black-and-white subgraph constructed on vertices of E_0) and consider a chain of homomorphisms

$\varphi_n : E_n \to E_{n+1}$ in $\mathcal{E}$. Due to connectedness of the graph E the union of edges of all elements of the chain will coincide with the set of edges of E. Moreover, if edge become black in the graph E_n from the chain, then it will be a black edge in any $E_N, N \geqslant n$. This means that E is indeed a limit of a chain E_n, φ_n. □

Now we construct a functor from the category $\mathcal{E}$ to abelian groups $\mathcal{AG}$. We associate to a finite black-and white graph $E \in \mathcal{E}$ the group $K_0(\tilde{\mathcal{O}}_E)$ with generators corresponding to all (black and white) edges $x_i \in E$ and relations $x_i = \sum_{y_i \in E^1} \lambda_{i,j} y_j$, for any black edge $x_i \in E$. Here y_j run over all edges, black and white of E, and $\lambda_i j$ is 1 if there is a path connecting directly x_i to y_j (they are adjacent in directed graph), except from the case when y_i is the inverse of x_i, in this case and all others $\lambda_{i,j}$ is zero.

Theorem 2.2. The map which sends an arbitrary black-and-white finite double graph $F \in \mathcal{E}$ to the group $K_0(\tilde{\mathcal{O}}_F)$ in $\mathcal{AG}$ can be extended to the functor $\mathcal{F}$ from $\mathcal{E}$ to $\mathcal{AG}$. This functor is continuous, i.e. it commutes with direct limits.

Proof. First we should show that defined above map which sends E to $K_0(\tilde{O}_E)$ is indeed a functor of those categories, i.e. morphisms of graphs are mapped into morphism of abelian groups. Let $\psi : E_n \to F_n$ be an elementary morphism of black-and-white graphs. We should ensure, that if we have any relation on elements of the group $\mathcal{F}(E)$: $P(x_1, ..., x_k) = 0,\ x_i = \mathcal{F}(y_i),\ y_i \in E$, then $P(\psi(y_1), ..., \psi(y_k)) = 0$. This is the case since the group $\mathcal{F}(F)$ has the set of generators, which includes generators of $\mathcal{F}(E)$ and amongst defining relations of $\mathcal{F}(F)$ we have got all defining relations of $\mathcal{F}(E)$. This is ensured, by the way we have constructed the graphs and associated the groups to them.

Now we can see that for an infinite graph E, $\mathcal{F}(E) = coker(Id - \Phi_E) = \mathcal{F}(\underline{\lim} E_n) = \underline{\lim} \mathcal{F}(E_n)$. This follows from the fact that defining relations of each $\mathcal{F}(E_n)$ is a subset of the defining relations of the group $\mathcal{F}(E) = coker(Id - \Phi_E)$. □

Combining this theorem with the proposition 2.1 we have the following.

Corollary 2.3. Any K_0 group of locally finite infinite graph E is a direct limitgroups corresponding to finite subgraphs from category $\mathcal{E}$.

3 K_0 calculations in the case of finite Betti number

After we have proved in previous sections, the existence of the direct limit in $\mathcal{AG}$ which express the value of K_0 for an arbitrary locally finite, infinite graph, we turn to concrete calculation in the case when the Betti number of that graph is finite.

First of all, we shall show that in any locally finite graph, we can perform any finite number of edge contractions, without changing K_0.

Theorem 3.1. *Let the graph E' be obtained from E by contraction of one non-loop edge x and its inverse $\bar{x}$. Then the groups $K_0(\mathcal{O}_E)$ and $K_0(\mathcal{O}'_E)$ are isomorphic.*

Proof. We will obtain the fact that the group K_0 is preserved under the edge contraction as a corollary of the following general lemma, which might be interesting in its own right.

Lemma 3.2. *Let G, H be abelian groups and $T : G \oplus H \to G \oplus H$ a homomorphism, such that $Tx - x \in G$ for any $x \in H$. Let $P : G \oplus H \to G$ be a homomorphism such that $P|_G = id_G$ and $Px = x - Tx$, $x \in H$.*

Then for $\tilde{T} : G \to G = P \circ T|_G$ the following is true:

$G \oplus H/T(G \oplus H) \simeq G/\tilde{T}(G)$

Proof. (of lemma 3.2).

Define a homomorphism $J : G/\tilde{T}(G) \to G \oplus H/T(G \oplus H)$ as follows:

$$J(u + \tilde{T}(G)) = u + T(G \oplus H).$$

The definition is correct, e.i. $\tilde{T}(G) \subseteq T(G \oplus H)$. Let $u \in G$ and $Tu = y + w, y \in H, w \in G$. Then $\tilde{T}u = P \circ Tu = P(y + w) = y - Tu + w = Tu - Ty \in T(G \oplus H)$.

The map J is injective. For $u \in G$, $u \in T(G \oplus H)$ implies $u \in \tilde{T}(G)$. Indeed, let $u = T(w + y) \in G$ for $w \in G, y \in H$. Since $u = Tw + Ty = Tw + y + Ty - y$, we can present Tw as $Tw = (u + (y - Ty)) - y$, and $u + (y - Ty) \in G, y \in H$. In 1. above we show that for $w \in G$, $\tilde{T}w = Tw = Th$, for h being an H-component of Tw: $Tw = w' + h$, $w' \in G, h \in H$. Due to the above presentation of Tw its H-component is $-y$, so we have: $\tilde{T}w = Tw + Ty = T(w + y)$, hence indeed $T(w + y) \in \tilde{T}(G)$.

The map J is surjective, i.e. $G + T(G \oplus H) = G \oplus H$. Indeed, for $x \in H$, there exists $u \in G : u = Tx - x$. Then for $w + x \in G \oplus H$, $w + x = Tx - u + w$, where $w = u \in G$ and $Tx \in T(G \oplus H)$.

□

Now to prove theorem 3.1 for the edge x and its inverse $\bar{x}$ apply the lemma 3.2 for direct sum of copies of $\mathbb{Z}$: $G = \mathbb{Z}^{|E^1\setminus\{x,\bar{x}\}|}$ and $H = \mathbb{Z}^2$. As an operator T ($\tilde{T}$) we should take $T = Id - \Phi_E$ ($\tilde{T} = Id - \Phi_{E'}$), where Φ defined by the formula (*).

□

We are now in a position to start the proof of the main theorem.

Theorem 3.3. *Let E be a locally finite connected graph with the finite first Betti number $\beta(E)$ and the brunching number $\gamma(E)$. Then $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)+\gamma(E)}$.*

Proof.
Type I. If the brunching number $\gamma(E)$ is finite, by a finite number of steps we can reduce our graph to the rose with $\beta(E)$ petals and $\gamma(E)$ outgoing simple infinite chains. In this case it is easy to calculate directly the group $K_0(\mathcal{O}_E) = coker(Id - \Phi)$, generated by relations readable from the graph. Indeed, let us denote variables corresponding to $\beta(E) = m$ petals (and their inverses) by $u_1, ..., u_m,\ \bar{u}_1, ..., \bar{u}_m$ and variables corresponding to $\gamma(E) = n$ edges outgoing directly from the vertex of the rose (and their inverses) by $x_1, ..., x_n,\ \bar{x}_1, ..., \bar{x}_n$. Next, edges (and their inverses) in each chain will be $x_i^{(1)}, ..., x_i^{(k)}, ...,\ \bar{x}_i^{(1)}, ..., \bar{x}_i^{(k)} ...,\ i = \overline{1,n}$. Then K_0 will be the quotient of the free group generated by the set $\Omega = \{x_i^{(k)},\ \bar{x}_i^{(k)}, i = \overline{1,n}, u_j, \bar{u}_j, j = \overline{1,m}\}$, subject to the relations defined by the formula $K_0(\mathcal{O}_E) = coker(Id - \Phi)$. For each edge $e \in E$ we will have one relation. Note that relation written for edges belonging to chains will give $x_i^{(1)} = x_i^{(2)} = ...,\ i = \overline{1,n}$. So after that we actually have a finite number of relations for variables $x_i, \bar{x}_i,\ i = \overline{1,n},\ u_j, \bar{u}_j,\ j = \overline{1,m}$. Namely,

$$\sum_{j\neq k}(u_j + \bar{u}_j) + \sum_{l=1}^{n} x_l = 0, \quad 1 \leqslant k \leqslant m,$$

$$\bar{x}_l = \sum_{j=1}^{m}(u_j + \bar{u}_j) + \sum_{r\neq l} x_r, \quad 1 \leqslant l \leqslant n,$$

Where the first group of relations correspond to petals and the second to edges outgoing (incoming) from (to) the rose. It is a compete set of defining relations for K_0 on the set of generators Ω.

For convenience let us denote by $w_j = u_j + \bar{u}_j$. Now write down the matrix of the above system of linear equations on variables $w_j, x_i, \bar{x}_i$, $j = \overline{1, m}$, $i = \overline{1, n}$.

$$\begin{pmatrix} 0 & \dots & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ & \ddots & & & \dots & & & \dots & \\ 1 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 1 & \dots & 1 & 0 & \dots & 1 & -1 & \dots & 0 \\ & \dots & & & \ddots & & & \ddots & \\ 1 & \dots & 1 & 1 & \dots & 0 & 0 & \dots & -1 \end{pmatrix}$$

By adding last n columns to the first m we can make zeros in the lower $n \times (m+n)$ block of the matrix. Then using the middle block of n columns we can transform the upper left $m \times (m+n)$ corner into

$$\begin{pmatrix} 1 & 0 & & \dots & & 0 & 0 & \dots & 0 \\ 0 & 1 & & \dots & & 0 & 0 & \dots & 0 \\ & & \ddots & & & & & & \\ 0 & 1 & & \dots & & 1 & 0 & \dots & 0 \end{pmatrix}$$

This shows that we have $m+n$ free variables: $u_1, ..., u_m, \bar{x}_1, ..., \bar{x}_n$. So we see, that $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)+\gamma(E)}$ in this case.

Type II. The second case is when the number $\gamma(E)$ is infinite. Here we can not write down a finite number of equations on the finite number of variables, which will define a group, but we can show what will be the system of free generators of the abelian group $K_0(\mathcal{O}_E)$. The group $K_0(\mathcal{O}_E)$ is defined by generators corresponding to all edges of the graph, consisting of one rose with $\beta(E)$ petals and finite number of outgoing infinite trees. The number of outgoing trees can not be infinite, because of locally finiteness condition.

Let us consider generators corresponding to petals of the rose: $u_1, ..., u_m$, $\bar{u}_1, ..., \bar{u}_m$ and edges coming out directly from the rose vertex: $x_1, ..., x_n$, $\bar{x}_1, ..., \bar{x}_n$. We have the following equations on them:

$$\sum_{k \neq j} (u_k + \bar{u}_k) + \sum_{l=1}^{n} x_l = 0, \quad 1 \leqslant j \leqslant m$$

$$\bar{x}_j = \sum_{k=1}^{m} (u_k + \bar{u}_k) + \sum_{l \neq r} x_l, \quad 1 \leqslant j \leqslant n$$

These are the same as above and gives us $n+m$ free variables: $u_1, ...u_m$, $x_1, ..., x_n$. Then consider for any piece of tree of the shape

$$\begin{array}{ccccc} & & & & b_1 \\ & & & \nearrow & \\ a & \longrightarrow & \bullet & & \vdots \\ & & & \searrow & \\ & & & & b_l \end{array}$$

equations which we have to write for edges a and b_j, $j = \overline{1,l}$, incoming for the fixed vertex (a) and those outgoing from it, which belongs to an infinite path. It is the following system.

$$a = b_1 + ... + b_l$$

$$\bar{b}_j = \bar{a} + b_1 + ... + \widehat{b_j} + ... + b_l$$

So, on such a step we get $l-1$ new free variables, corresponding to $l-1$ new infinite pathes along the graph, we got in this vertex. If we sum up all new free variables, we have in all vertices of outgoing trees, we arrive to $\gamma(E)$ additional variables. Note that again, if an outgoing chains are finite, then variables corresponding to their edges are just zero. So, we see that in this case also $K_0(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)+\gamma(E)}$, and here it is a direct sum of countably infinite number of copies of $\mathbb{Z}$, and the set enumerating these copies is in $1-1$ correspondence with infinite outgoing pathes. By this the proof of the theorem is completed.

$\square$

4 The Whitehead group expressed via the first Betti number

In the original paper due to Cuntz and Krieger [4] it was shown that K_0 and K_1 of the Cuntz-Krieger C^*- algebra $\mathcal{O}_A$, associated to any finite 0-1 matrix A are, respectively, co-kernel and kernel of the map $(Id - A^t) : Z^n \to Z^n$.

This fact was later generalized in [6] to the Cuntz-Krieger C^*- algebra $\mathcal{O}_A$, associated in the same way to infinite 0-1 matrix A, with finite number of 'ones' in any row. The graph algebra we consider, as it is mentioned in Introduction, is a Cuntz-Krieger algebra associated to an infinite matrix,

constructed from the graph by certain rules. So this result is applicable here and making use of this we will prove the following.

Theorem 4.1. *Let E be a locally finite connected graph with the finite first Betti number $\beta(E)$, and $\mathcal{O}_E$ is an associated C^*- algebra (via Bass-Hashimoto operator). Then $K_1(\mathcal{O}_E) = \mathbb{Z}^{\beta(E)}$.*

Proof. The proof will be divided into several steps.

First of all, we shall show that in any locally finite graph, we can perform any finite number of edge contractions, without changing K_1.

Theorem 4.2. *Let the graph E' be obtained from E by contraction of one non-loop edge x and its inverse $\bar{x}$. Then the groups $K_1(\mathcal{O}_E)$ and $K_1(\mathcal{O}_{E'})$ are isomorphic.*

Proof. Let us ensure the following fact of linear algebra.

Lemma 4.3. *Let G, H be abelian groups and $T : G \oplus H \to G \oplus H$ a homomorphism, such that $Tx - x \in G$ for any $x \in H$. Let $P : G \oplus H \to G$ be a homomorphism such that $P|_G = id_G$ and $Px = x - Tx$, $x \in H$.*

Then for $\tilde{T} : G \to G = P \circ T|_G$ the following is true: $\mathrm{Ker}T \simeq \mathrm{Ker}\tilde{T}$.

Proof. (of lemma 4.3)

Take an element $u + y \in \mathrm{Ker}T$, with $u \in G$, $y \in H$. Let $Tu = w + x$, where $w \in G$, $x \in H$. Then $\tilde{T}u = x - Tx + w = Tu - Tx$, so $Tu = \tilde{T}u + Tx$. Now substituting that to $T(u+y) = 0$, we have $0 = T(u+y) = \tilde{T}u + T(x+y)$. From this we see first that $\tilde{T}u = -T(x + y)$.

Denote G and H components of an element $r \in G \oplus H$ by r_G and r_H respectively, so $r = r_G + r_H$, for $r_G \in G$ and $r_H \in H$. Now comparing G and H components of left and right hand side of $\tilde{T}u = -T(x+y) = -(x+y) - g_{x+y}$, we have that $x + y = 0$. Therefore $\tilde{T}u = -T(u + y)$ and $u + y \in \mathrm{Ker}T$ iff $u \in \mathrm{Ker}\tilde{T}$. □

The proof of the theorem 4.3 will follow as a corollary from this lemma if we put $G = \mathbb{Z}^{|E^1 \setminus \{x, \bar{x}\}|}$ and $H = \mathbb{Z}^2$. As an operator T ($\tilde{T}$) we should take $T = Id - \Phi_E$ ($\tilde{T} = Id - \Phi_{E'}$), where Φ defined by the formula (*).

□

Theorem 4.2 from the first step, allows us to reduce calculation of $K_1(\mathcal{O}_E)$, where E is a locally finite graph with Betti number $\beta(E)$ to the K_1 for the graph Γ, which is a rose with $\beta(E) = \beta(\Gamma)$ petals and a tree, rooted in the

vertex of the rose, with finite number of branches outgoing from each vertex, due to locally finiteness condition.

Now the second step in the calculation of the Whitehead group will be a calculation for the graph Γ. We need to calculate $K_1(\mathcal{O}_\Gamma) = \operatorname{Ker} T_\Gamma = \operatorname{Ker}(Id - \Phi_\Gamma)$, where Φ_Γ defined by formula (*).

For the graph Γ we can present the set of all edges as a disjoint union of three sets:

$$\Gamma^1 = \Gamma^\uparrow \sqcup \Gamma^\downarrow \sqcup R,$$

where $\Gamma^\uparrow$ is the set of edges from the tree, directed towards the rose, $\Gamma^\downarrow$ is the set of edges of the tree directed off the rose and R is the set of petals of the rose.

Let $\xi \in \operatorname{Ker} T_\Gamma$, $\xi = \sum_{e\in\Gamma_\xi} m_e e$, where Γ_ξ is a finite set of edges and $m_e \in \mathbb{Z} \setminus \{0\}$.

Let us show first that the following is true.

Lemma 4.4. *The set Γ_ξ does not contain tree edges in the direction towards the rose: $\Gamma_\xi \cap \Gamma_\uparrow = \emptyset$.*

Proof. Consider projection $\pi : \mathbb{Z}^{(\Gamma)} \longrightarrow \mathbb{Z}^{(\Gamma)}/\mathbb{Z}^{(\Gamma')}$, where $\Gamma' = \Gamma^\downarrow \sqcup R$, then the composition of our initial map T with π denote by T':

$$\mathbb{Z}^{(\Gamma)} \xrightarrow{T} \mathbb{Z}^{(\Gamma)} \xrightarrow{\pi} \mathbb{Z}^{(\Gamma)}/\mathbb{Z}^{(\Gamma')} \simeq \mathbb{Z}^{(\Gamma^\uparrow)})$$

Then $T'/\mathbb{Z}^{(\Gamma')} = 0$ and $T'e = e + f$, where f consists of edges which are higher in the tree (=closer to the rose) then e.

Suppose $\Gamma_\xi \cap \Gamma^\uparrow \neq \emptyset$. Consider $g \in \Gamma_\xi$ farthest away from the rose, $\xi = mg + \tilde{g}$, $m \in \mathbb{Z} \setminus \{0\}$.

Then

$$T'\xi = mg + f + T'(\tilde{g}).$$

Here f consists from terms corresponding to the edges, closer to the rose then g. $T'(\tilde{g})$ consists from terms corresponding to the edges, closer to the rose then $\tilde{g}$, which are in a turn closer then g. This means that the term mg can not cancel, and $T'\xi \neq 0$, hence $T\xi \neq 0$. We arrive to a contradiction. $\square$

Lemma 4.5. *The set Γ_ξ does not contain tree edges in the direction off the rose: $\Gamma_\xi \cap \Gamma_\downarrow = \emptyset$.*

Proof. Assume $\Gamma_\xi \cap \Gamma^{\downarrow} \neq \emptyset$, and take $h \in \Gamma_\xi \cap \Gamma^{\downarrow}$, farthest away from the rose. Then

$$Th = h - (h_1 + h_2 + ...),$$

where all h_i are further away then h from the rose. Then

$$\xi = mh + \sum h_i g_i,$$

$m \in \mathbb{Z} \setminus \{0\}$, g_i closer then h to the rose by the choice of h and $g_i \notin \Gamma^{\uparrow}$ by the previous lemma. Therefore

$$T\xi = mh - m(h_1 + h_2 + ...) + T(\sum h_i g_i),$$

and the furthest from the rose edge, which $T(\sum h_i g_i)$ could contain is h. This means that the term mh_1 could not be canceled, $T\xi \neq 0$ and we arrive to a contradiction. $\square$

Now after above two lemmas we are left with the only possibility, that

$$\Gamma_\xi \subseteq R = \{u_1, ..., u_n, \bar{u}_1, ..., \bar{u}_n\}.$$

Let $\xi = \sum_{j=1}^{n}(m_j u_j + n_j \bar{u}_j)$. We know that

$$T\bar{u}_j = Tu_j = -\sum_{k \neq j}(u_k + \bar{u}_k) + \sum_{=1}^{s} x_l = w - (u_j + \bar{u}_j),$$

where for convenience we denote by $w = -\sum_{k=1}^{n} + \sum_{l=1}^{s} x_l$. Then

$$T\xi = (\sum_{j=1}^{n}(m_j + n_j))w - \sum_{j=1}^{n}(m_j + n_j)(u_j + \bar{u}_j) = 0.$$

Since all x_l appear in w with coefficient 1 they can disappear only if $\sum_{j=1}^{n}(m_j + n_j) = 0$. Hence

$$T\xi = -\sum_{j=1}^{n}(m_j + n_j)(u_j + \bar{u}_j) = 0.$$

Since each u_j appears in one term only, in order sum to be zero, we should have $m_i + n_i = 0$ for all j. This means $\xi \in \mathrm{Ker}T \iff \xi = \sum_{j=1}^{n} m_j(u_j - \bar{u}_j)$, $m_j \in \mathbb{Z}$, thus $\mathrm{Ker}T = \mathbb{Z}^n$, where $n = \beta(\Gamma)$. This completes the proof of the theorem 4.1.

$\square$

Remark This result shows that the statement that K_1 is a torsion free part of K_0, which holds in case of finite graphs (see [10] or [3]) is false in the setting of infinite graphs.

Indeed, let E be locally finite infinite graph with finite first Betti number $\beta(E)$ and infinite brunching number $\gamma(E)$. Then $K_0(\mathcal{O}_\mathcal{E}) \simeq \mathbb{Z}^\infty$ and $K_1(\mathcal{O}_\mathcal{E}) \simeq \mathbb{Z}^{\beta(\mathcal{E})}$, so $K_1(\mathcal{O}_\mathcal{E})$ is not isomorphic to the torsion part of $K_0(\mathcal{O}_\mathcal{E})$.

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